(DM 01)

M.Sc. (Previous) Mathematics DEGREE EXAMINATION, MAY 2007

Paper I - **ALGEBRA**

Time : Three hours | Maximum : 100 marks

Answer any FIVE questions.
All questions carry equal marks.

1. (a) If ϕ is a homomorphism of a group *G* into group $\overline{G}$ with Kernel *K*, then prove that *K* is a normal subgroup of *G*.

   (b) If *G* is an abelian group of order $o(G)$ and if *p* is prime number such that $P^{\alpha} / o(G), p^{\alpha+1}\, o(G)$ then prove that *G* has a subgroup of order P^{α} .

2. (a) Show that every group is isomorphic to a subgroup of A(S) for some appropriate S.

   (b) (i) Show that every permutation is a product of transpositions and

   (ii) If a permutation $\sigma \in S_n$ is a product of *r* transpositions and also a product of s transpositions, then show that either *r* and s are both even or both odd.

3. (a) Prove that the number of *p*-sylow subgroups in *G*, for a given prime, if of the form 1+*Kp*.

   (b) Prove that two abelian groups of order p^n are isomorphic if and only if they have the same invariants.

4. (a) Prove that every integral domain can be embedded in a field.

   (b) Let R be a Euclidean ring. Prove that an element $a \in R$ is a prime element of *R* if and only if the principal ideal (□) is a maximal ideal of *R*.

5. (a) If R is a unique factorization domain, then prove that the polynomial ring $R[x]$ is also a unique factorization domain.

   (b) Prove that the ring $J(i)$ of Gaussian integers is a Euclidean ring.

6. (a) If *K* is an extension of field *F*, then prove that the element $a \in K$ is algebraic over *F* if and only if *F*(*a*) is a finite extension of *F*.

   (b) Prove that a polynomial of degree *n* over *a* field *F* can have atmost *n* roots in any extension field.

7. (a) Prove that a polynomial of degree *n* over *a* field can have atmost *n* roots in any extension field.

   (b) Prove that it is impossible to trisect 60^0, by straight edge and compass alone.

8. (a) State and prove the Fundamental theorem of Galo's theory.

(b) Show that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

9. (a) Define a lattice, a modular lattice and a distributive lattice. Prove that every distributive lattice is modular. Is the converse of this statement true ? Justify your answer.

(b) Show that

(i) Every chain is a distributive lattice

(ii) $(a*b)' = a' \oplus b' \; and \; (a+b)' = a' * b'$ hold in a complemented distributive lattice.

10 (a) If *L* is a semi-modular lattice of finite length and $a, b \in L$ with *a* < *b*, show that any two connected chains from *a* to *b* in *L* have the same lenght.

(b) Define a Boolean algebra and a Boolean ring. Let $(B, *, \oplus, ', 0, 1)$ be a Boolean algebra. Define the operations + and . on the elements of *B* by.

$$a + b = (a * b') \oplus (a' * b)$$

$$a.b = a * b$$

Show that $(B, +, \bullet, 1)$ is a Boolean ring with identity 1.

(DM 02)

M.Sc. (Previous) Mathematics DEGREE EXAMINATION, MAY 2007

Paper II - **ANALYSIS**

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) (i) Prove that every infinite subset of a countable set is countable.

(ii) Show that the set of all rational numbers is countable. Is the set of all irrational real numbers countable ? Justify your answer.

(b) Let X be a metric space and $E \subset X$.. Then show that

(i) $\overline{E}$ is closed

(ii) $E = \overline{E}$ if and only if *E* is closed

(iii) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

2. (a) When do you say that a subset *K* of a metric space *X* to the compact ? Show that every *K* cell is compact.

   (b) (i) Construct a bounded set of real numbers with exactly three limit points.

   (ii) Construct a compact set of real numbers whose limit points form a countable set.

3. (a) Let *X* be a metric space. Then show that.

   (i) Every convergent sequence in *X* is a cauchy sequence

   (ii) if X is compact and if $\{P_n\}$ is a Cauchy sequence in X, then $\{P_n\}$ converges to some point of *X*.

   (iii) In R^k, every Cauchy sequence converges.

   (b) Show that the product of two convergent series also converges, if atleast one of the two series converges absolutely.

4. (a) Prove that a mapping *f* of a metric space *X* into a metric space *Y* is continuous on *X* if and only if $f^{-1}(V)$ is open in *X* for every open set *V* in *Y*.

   (b) Let f be a continuous mapping of a compact metric space *X* into a metric space *Y.* Then show that *f* is uniformly continuous on *X*.

5. (a) When do you say that a bounded real function f is *R-S* integrable over [a,b]? If f is monotonic on [a,b] and α is continuous on [a,b] then, prove that $f \in R(\alpha)$.

   (b) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on [a,b], then prove that

   (i) $fg \in R(\alpha)$ and

   (ii) $|f| \in R(\alpha)\, and \left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha.$

6. (a) Assume that α increases monotoncially and $\alpha' \in R$ on [a,b]. Let f be a bounded real function on [a,b]. Then prove that $f \in R(\alpha)$ if and only if $f\alpha' \in R$ and in that case $\int_a^b f\, d\,\alpha = \int_a^b f(x)\alpha'(x)dx.$

   (b) State and prove the fundamental theorem of calculus.

7. (a) If $\{f_n\}$ is a sequence of continuous functions on *E* and $f_n \to f$ uniformly on *E*, then show that *f* is continuous on *E*, what can you say about the converse of this result ? Justify your answer.

(b) State and prove the Cauchy criterion for uniform convergence a sequence of functions $\{f_n\}$ defined on *E*.

8. (a) If *f* is a continuous complex function on [a,b] then show that there exists a sequence of polynomials P_n such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on [a,b]. If *f* is real, then P_n may be taken real.

9. (a) Define the outer measure $\mu^*(E)$ of a set $E \subset R^p$, with the usual notation, show that

   (i) For every $A \in \xi, \mu^*(A) = \mu(A)$.

   (ii) If $E = \bigcup_1^\infty E_n, then\ \mu^*(E) \le \sum_{n=1}^\infty \mu^*(E_n)$.

   (b) State and prove Lebergue dominated convergence theorem.

10 (a) Define a measurable function. If f and g are measurable, then show that (i) *f*+*g* and *fg* are measurable. (ii) $|f|$ is measurable.

   (b) State and prove Riesz - Fischer theorem.

(DM 03)

M.Sc. (Previous) Mathematics DEGREE EXAMINATION, MAY 2007

Paper III - **COMPLEX ANALYSIS AND SPECIAL FUNCTIONS AND PARTIAL DIFFERENTIAL EQUATIONS**

Time : Three hours Maximum : 100 marks

Answer any FIVE questions choosing atleast TWO questions from each Part.

PART - A

1. (a) (i) Show that $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$

   (ii) Express $P(x) = x^4 + 2x^3 + 2x^2 - x + 3$ in terms of Legendre's polynomials.

   (b) Prove that $n P_n(x) = x P'_{n-1}(x)$.

2. (a) Find the solution of the Bessel's differential equation of order *n* of the first kind, *n* being non-negative constant.

(b) (i) Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

(ii) Show that $xJ_n^{'}(x) = -n J_n(x) + x J_{n-1}(x)$.

3. (a) Prove that $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0, \quad if\ m \neq n$

$$= \frac{2}{2n+1}, \ if\ m = n$$

with the usual notation.

(b) Show that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) + \ldots\ldots and$$
$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) + \ldots\ldots$$

4. (a) Verify that the equation $yz\,dx + \left(x^2y - zx\right)dy + \left(x^2z - xy\right)dz = 0$ is integrable and find its primitive.

(b) Solve $\left(D^2 - 2DD' + D'^2\right)Z = e^{x+2y}$ with the usual notation.

5. (a) Find a complete integral of the equation $2zx - px^2 - 2qxy + pq = 0$ using Chorpits method.

(b) Solve $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$ by Monge's method.

PART B

6. (a) For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, define the number R, $0 \le R \le \infty\, by \frac{1}{R} = \lim\sup |a_n|^{1/n}$. Then prove that (i) if $|z-a| < R$, the series converges absolutely (ii) if $|z-a| > R$, the series diverges (iii) if $0 < r < R$ then the series converges uniformly on $\{z : |z| \le r\}$.

(b) Show that $f(z) = \overline{z}$ is not analytic.

7. (a) Define mobius transformation. Show that a mobius transformation takes circles onto circles.

(b) (i) State and prove the symmetry principle.

(ii) Show that $u(x,y)=e^x \cos y \,\forall\, (x,y) \in R^2$ is harmonic. Find an analytic function *f* whose real part is *u*.

8. (a) Let γ be a closed polygon : $[1-i, 1+i, -1+i, -1-i, 1-i]$. Then find $\int_\gamma \frac{1}{2} dz$.

(b) State and prove Cauchy's integral formula first version.

9. (a) State and prove the liouvilles theorem.

(b) Suppose that υ is a connected open set and *f(z)* is an analytic function such that $|f(z)| = c$, a constant for $z = \upsilon$. Then show that *f(z)* is a constant.

10 (a) State and prove the Residue theorem.

(b) Evaluate $\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta}$ using Residue theorem.

(DM 04)

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Paper IV - **THEORY OF ORDINARY DIFFERENTIAL EQUATIONS**

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Prove that there exist *n* linearly independent solutions of

$$Ly = y^{(n)} + a_1(x)\,y^{(n-1)} + \cdots + a_n(x)\,y = 0.$$

Where $a_i(x), i = 1, 2 \cdots n$ are continuous function on *I* ?

(b) Find two linearly independent solutions of the equations

$$(3x-1)y'' + (9x-3)\,y' - 9y = 0 \; for \; x > 1/3.$$

2. (a) One solution of $x^2y''-xy'+y=0,(x>0)\, is\, \phi,(x)=x.$

Find the solution ψ of the equation

$x^2y''-xy'+y=x^2.$

Satisfying $\psi(1)=0,\psi^1(1)=0$

(b) Find two linearly independent power series solution of the equation $y''+3x^2y'-xy=0.$

3. (a) Show that the solution $\phi\, of\; y^1=y^2$ which passess through the points (x_0,y_0) is given by

$$\phi(x)=\frac{y_0}{1-y_0(x-x_0)}.$$

(b) Find an integrating factor for the equation cos *x* cos *y dx-2* sin *x* sin *y dy = 0* and solve it.

4. State and prove the existence of solution of the initial value problem $y'=f(x,y),y(x_0)=y_0$ on *I*.

5. (a) Suppose that *f* is a continuous function on an interval $|x-x_0|\le a.$ show that ϕ is a solution of initial value problem.

$y'=f(x),y(x_0)=\alpha,y'(x_0)=\beta$

if and only if it is a solution of integral equation.

$$\phi(x)=\alpha+\beta(x-x_0)+\int_{x_0}^{x}(x-t)\,f(t)\,dt.$$

(b) Find a solution ϕ of $y''=-1/2y^2$ satisfying $\phi(0)=0,\ \phi^1(0)=-1$.

6. (a) Suppose *f* is a vector valued function defined for (x,y) on a set *S* of the form

$|\,x-x_0\,|\le a,|\,y-y_0|\le b,(a,b>0)$ or of the forms.

$|\,x-x_0\,|\le a,|y|<\infty\,(a>0).\ If\ \partial f\left|\partial y_k^{(k=1,2,\cdots,n)}\right.$ exists, is continuous on *S*, and there is a constant

K > 0 such that $\left|\frac{\partial f}{\partial y_k}(x,y)\right|\le K\,(K=1,2,\cdots,n)$ for all (x,y) in *S*. Prove that *f* satisfies a Lipschitz condition on *S* with Lipschitz constant *K.*

(b) Find a solution ϕ of the system

$y_1^1 = y_2$

$y_2^1 = 6y_1 + y_2$, satisfying $\phi(0) = (1, -1)$.

7. (a) Compute the Green's function for the differential equation $y'' - 5y' + 6y = 0$.

(b) Find the general solution of $y'' - 3y' + 2y = x \, (-\infty < x < \infty)$ by computing the particular solution using Green's theorem.

8. (a) Study the solutions of the Riccatic equation $Z^1 + Z - e^x Z^2 - e^{-x} = 0$.

(b) Find four distinct solutions for the differential equation $Z^1 + Z^2 - 1 = 0$.

9. (a) Show that the equation $x'' + a(t)x' + b(t)x = 0$

Can be transformed into self adjoint equation by multiplying the equation by integrating factor.

(b) Prove that the Euler's equation $x'' + \frac{k}{t^2} x = 0$

(i) is oscillatory if *K > 1/4* and

(ii) is non-oscillatory if $K \le 1/4$

10 (a) State and prove Sturm's comparison theorem.

(b) Let *r (t)* be a positive continuous function and let *m* be a real number. Show that the equation

$x'' + \left(m^2 + r(t)\right) x = 0, \; t \ge 0$ is oscillatory.

M.Sc. (Final) Mathematics DEGREE EXAMINATION, MAY 2007

Paper I - **TOPOLOGY AND FUNCTIONAL ANALYSIS**

Time : Three hours Maximum : 100 marks

Answer any FIVE questions choosing at least TWO from each of Part A and Part B.

1. (a) If *X* is a nonempty set and c is a self-map on *X* satisfying Kuratowski closure axioms, prove that $Y=\{X \setminus A : A \subseteq X \; and \; c(A)=A\}$ is a topology on *X* and that the closure of any subset *A* of *X* with respect to Y is *c(A).*

   (b) If *X* is a nonempty set, prove tht the family of all topologies on *X* is a complete lattice with respect to the relation "is weaker than".

2. Prove that on topological space is compact if every subbasic open cover has a finite subcover.

3. (a) State and prove Tychonoff's theorem.

   (b) Prove that every sequentially compact metric space is totally bounded.

4. (a) Prove that every compact Haushorff space is normal.

   (b) State and prove Urysohn's imbedding theorem.

5. (a) Prove that the product of any nonempty class of connected spaces is connected.

   (b) Show that the union of any non-empty class of connected subspaces of a topological space, each pair of which intersects is connected.

PART B

6. State Hahn-Banach theorem and the lemma associated with it and prove both of them.

7. State the open mapping theorem and the lemma associated with it and prove both of them.

8. (a) If the norm on a normed linear space satisfies the parallelogram law, prove that it can be derived from an inner product.

   (b) State and prove Bessel's inequality.

9. (a) State and prove Riesz representation theorem for bounded linear functionals on a Hilbert space.

   (b) Prove that a Hilbert space is reflexive.

10 Let *T* be an operator on a Hilbert space *H.* Then prove the following statements.

(a) T is self-adjoint $\Leftrightarrow$ (Tx, r) is real for all x in X.

(b) T is normal $\Leftrightarrow$ the real and imaginary parts of T commute.

(c) T is unitary $\Leftrightarrow$ T is an isometric isomorphism of H into itself.

(DM 22)

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Paper II - **MEASURE AND INTEGRATION**

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.
All questions carry equal marks.

1. (a) Define F_σ and G_δ sets. Show that a countable set and an open set are F_σ sets.

   (b) Define a partial ordering of a set and state Hausdorff maximum principle.

   (c) Define the extended real number system.

2. (a) Define the Lebesgue measure of a set. Show that if E_1 and E_2 are measurable then $E_1 \cup E_2$ is also measurable.

   (b) If f and g are two measurable, real valued functions defined on the same domain, then show that $f+g$ and fg are also measurable.

3. (a) If E has the outer measure zero, then show that E is a measurable set. Show also that every subset of E is measurable.

   (b) Show that a continuous function defined on a measurable set E, is measurable.

4. (a) If $\int_E f(x)dx = 0$ and $f(x) \geq 0$, on a measurable set E, then show that $f=0$, a.e on E.

   (b) State and prove bounded convergence theorem.

5. (a) State and prove Lebesgue convergence theorem.

   (b) If a sequence $\{f_n\}$ converges is measure to f, then show that the limit function f is unique a.e.

6. (a) If *f* is integrable on [a, b] then show that the function *F*, defined by $F(x) = \int_a^x f(t)dt,$ is continuous function of bounded variation on [a, b].

(b) Let *f* be an integrable function on [a,b] and suppose that $F(x) = F(a) + \int_a^x f(t)dt.$ Then show that *F'(x) = f(x)* for almost all x in [a, b].

7. (a) State and prove Minkowski inequality.

(b) Let *g* be an integrable function on [0,1], and suppose that there is a constant *M* such that $\left|\int fg\right| \leq M\|f\|_p$ for all bounded measurable functions *f*. Then show that $g \in L^q$ and $\|g\|_q \leq M$, where $1 < p < \infty \; and \; \frac{1}{p} + \frac{1}{q} = 1..$

8. (a) Define a measurable space (X, B) and the measure μ on a measurable space (X, B). If $E_i \in B$ and $E_i \supset E_{i+1}$ with $\mu E_1 < \infty,$ then show that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \underset{n\to\infty}{Lt}\, \mu E_n.$

(b) Define

(i) a signed measure on the measurable space (X, B) and

(ii) a positive set with respect to a signed measure. Prove that the union of a countable collection of positive sets is positive.

9. (a) State and prove Hahn-Decomposition theorem.

(b) Prove with the usual notation that if μ and γ are positive measures, then they are mutually singular if and only if $\mu \cap \gamma = 0 \; where \; \mu \cap \gamma = \frac{1}{2}(\mu + \gamma - |\mu - \gamma|).$

10 (a) With the usual notation derive μ^* - measurable set. Prove that the set function μ^*, is an outer measure.

(b) If $B \in A$, algebra of sets, then show that *B* is measurable with respect to μ^*.

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Paper III - **ANALYTICAL NUMBER THEORY AND GRAPH THEORY**

Time : Three hours Maximum : 100 marks

Answer any FIVE out of the given TEN questions
selecting atleast TWO from each Part.

1. (a) State and prove Euler's summation formula.

   (b) If $x \geq 1$, show that $\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right)$ where *C* is Euler's constant.

2. (a) If $x \geq 1$ and α > 0, $\alpha \neq 1$, show that $\sum_{n \leq x} \sigma_\alpha(n) = \frac{\varsigma(\alpha+1)}{\alpha+1} x^{\alpha+1} + D\left|x^\beta\right|$, where $\beta = \max\{1, \alpha\}$.

   (b) Show that the set of lattice points visible from the origin has density $6/\pi^2$.

3. (a) For x > 0, show that $0 \leq \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}$.

   (b) For $x \geq 2$, show that $\theta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} dt.$

4. (a) For every integer $n \geq 2$, show that $\frac{n}{6\log n} < \pi(n) < 6\frac{n}{\log n}$.

PART B

5. (a) Show that a graph *G* is disconnected if and only if its vertex set *V* can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in *G* whose one end vertex is in subset V_1 and the other in subset V_2.

   (b) If a graph has exactly two vertices of odd degree, show that there must be a path joining these two vertices.

6. (a) In a connected graph *G* with exactly 2k odd vertices, show that there exist *k* edge-disjoint subgraphs such that they together contain all edges of *G* and that each is a universal graph.

   (b) Show that a connected graph *G* is an Euler graph if and only if it can be decomposed into circuits.

7. (a) If in a graph G there is one and only one path between every pair of vertices, show that G is a tree.

(b) Show that a graph G with n vertices, $n-1$ edges and no circuits is connected.

8. (a) Show that energy circuit has an even number of edges in common with any cut-set.

(b) Show that the vertex connectivity of any graph G can never exceed the edge connectivity of G.

9. (a) Show that a connected planar graph with n vertices and e edges has $e-n+2$ regions.

(b) In any simple, connected planar graph with f regions, n vertices and e edges (e > 2), show that the following inequalities must hold :

(i) $e \geq \frac{3}{2} f$ and (ii) $e \geq 3n-6$

10 (a) Show that the set consisting of all the circuits and the edge-disjoint unions of circuits (including the null set ϕ) in a graph G is an abelian group under the ring-sum operation $\oplus$.

(b) Show that the set consisting of all the cut-sets and the edge-disjoint unions of cut-sets (including the null set ϕ) in a graph G is an abelian group under the ring-sum operation.

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Paper IV - **RINGS AND MODULES**

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.
all questions carry equal marks.

1. (a) Define a Boolean algebra. Prove that a Boolean algebra becomes a complemented distributive lattice by defining $a \wedge b = (a' v\, b')\, and\, 1 = 0'$

(b) Let R, S be two rings. If *f:* $f : R \to S$ is a homomorphism, then prove that there exists a congruence relation θ on R, an epimorphism $\pi : R \to R/\theta$ and a monomorphism $g : R/\theta \to S$ such that $f = g \circ \pi$.

2. (a) If R is a ring, prove that the following statements are equivalent.

(i) *R* is isomorphic to a finite direct product of Rings $R_i\,(1 \le i \le x)$..

(ii) There exist central idempotents $e_i \in R(1 \le i \le n)$ such that $\sum_{i=1}^{n} e_i = 1$ and $e_i R$ is isomorphic to R_i for $1 \le i \le n.$

(iii) *R* is a finite direct sum of ideals K_i isomorphic to $R_i\,(1 \le i \le x)$.

(b) Let *C* be a submodule of A_R. Prove that every submodule of *A/C* has the form *B/C* for some submodule *B* to A_R with $C \subseteq B \subseteq A$. Also prove that *A/B* is isomorphic to *(A/C) / (B/C).*

3. (a) Let *B* be a submodule of A_R. Prove that *A* is Noetherian if and only if *B* and *A/B* are Noetherian.

(b) If *e* is a central idempotent in a ring *R*, then prove that *e R* is indecomposable if and only if *e* is an atom of the Boolean algebra *B(R)* of all central idempotents of *R.*

4. (a) Prove that every ring is a subdirect product of subdirectly irreducible rings.

(b) Prove that every commutative regular ring is semi primitive.

5. (a) If *R* is a commutative ring, then prove that its complete ring of Quotients *Q(R)* is rationally complete.

(b) If *R* is a commutative ring, then prove that *Q(R)* is regular if and only if *R* is semi prime.

6. (a) State and prove Jacobson - Density theorem.

(b) Define the concept of a prime ideal. Prove that every primitive ideal of a ring is prime.

7. (a) Let *R* be a ring. Prove that the following conditions are equivalent

(i) 0 is the only nilpotent ideal of *R*

(ii) 0 is an intersection of prime ideals

(iii) For any ideals *A* and *B* of *R, AB = 0* $\Rightarrow A \wedge B = 0$.

(b) If *B* is a submodule of an *R*-module *A* and *C* is maximal among the submodules of *A* such that $B \cap C = \{0\}$, then prove that *B + C* is large.

8. (a) If *k* is a minimal right ideal of a ring *R,* then prove that either $k^2 = 0 \; or \; k = eR$ for some idempotent $e \in k$.

(b) If *R* is a right Artinian ring, then prove that any right *R*-module is Noetherian if and only if it is Artinian.

9. (a) Prove that every free module is projective. Give an example of a module which is projective but not free.

   (b) Prove that on R-module is projective if and only if R is completely reducible.

10 (a) Prove that an abelian group is injective if and only if it is divisible.

   (b) Prove that a module M is injective if and only if M has no proper essential extension.